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n-polar Z-hesitant Complementary Fuzzy Soft Set in BCK/BCI-Algebras

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Abstract

This paper introduces an innovative concept known as n-polar Z-hesitant Anti-Fuzzy Soft Sets (MZHAFSs) within the framework of BCK/BCI-algebras. Soft set theory originates in the captivating field of fuzzy set theory. Our approach is a harmonious synthesis of n-polar anti-fuzzy set theory, soft set models, and Z-hesitant anti-fuzzy sets, skillfully applied within the framework of BCK/BCI-algebras. This effort leads to the introduction of a new variant of fuzzy sets termed MZHAFSs (n-polar Z-hesitant anti-fuzzy soft sets) in the context of BCK/BCI-algebras. Additionally, we elucidate the concept of n-polar Z-hesitant anti-fuzzy soft sets to provide a comprehensive understanding. Furthermore, we introduce and define various related concepts, including n-polar Z-hesitant anti-fuzzy soft subalgebras, n-polar Z-hesitant anti-fuzzy soft ideals, n-polar Z-hesitant anti-fuzzy soft closed ideals, and n-polar Z-hesitant anti-fuzzy soft commutative ideals, and establish meaningful connections between them. We also present and rigorously prove several theorems that are pertinent to these newly introduced notions.

Keywords: n-polar Z-hesitant anti-fuzzy sets; fuzzy logic; fuzzy control.

1 Introduction

In disciplines like economics, engineering, environmental science, social science, and management science, intricate challenges are a common occurrence. These challenges often manifest with features such as uncertainty, imprecision, and vagueness. Conventional methods struggle to effectively address these issues due to the diverse nature of uncertainties involved. Additionally, despite the availability of mathematical theories like probability, fuzzy set (FS), interval mathematics, and rough set as prospective approaches for addressing uncertainty, it is worth noting that Molodtsov Molodtsov introduced the concept of soft sets theory in 1999 as a novel mathematical framework designed to manage uncertainty [29]. In response to these intricate challenges, Molodtsov introduced a pioneering mathematical tool known as soft set theory, designed to grapple with uncertainties and imprecisions. This theory has demonstrated its efficacy in various practical applications, including decision-making, measurement theory, and game theory [28]. Importantly, the versatility of the soft set model allows it to seamlessly integrate with other mathematical frameworks. Maji and colleagues pioneered the fusion of fundamental concepts from fuzzy sets (FSs) and soft sets, introducing the innovative notion of fuzzy soft (FSo) sets [27], and some proprities fuzzy soft sets in [4]. It's worth noting that the formulation of FSs traces back to Zadeh's initial work in 1965 [35].

In recent years, research in the field of fuzzy sets and their applications has seen remarkable growth, with several pivotal contributions that have expanded our understanding of these mathematical structures. Notable among these works, are Q-fuzzy soft set [3], New types of hesitant fuzzy soft set ideals in BCK-algebras [8], hesitant anti-fuzzy soft set in BCK-algebras [9]. In addition to these, the literature is enriched with various other contributions, including a decision-making approach based on a multi Q-hesitant fuzzy soft multi-granulation rough model by Alsager et al. [7].

In 2010, Torra introduced a method for expressing people's hesitancy in everyday situations. Hesitant FSs have found invaluable utility in decision-making scenarios, providing a precise avenue for capturing uncertainty through the viewpoints of decision-makers [32]. Bridging the realms of classical soft sets and hesitant FSs. In 2013, the concept of hesitant fuzzy soft sets was first proposed by Babitha and John [11]. On the contrary, back in 1966, Imai and Iseki introduced the concepts of BCI and BCK algebras, contributing to the field of universal algebra with structures that model specific aspects of propositional calculus, specifically implication. These structures are known as BCI and BCK logics.

In 2014, Chen and their team delved further into the realm of bipolar FSs, introducing the novel concept of n-polar FSs. Their research brought to light a fundamental mathematical equivalence between bipolar FSs and 2-polar FSs [12]. Additionally, Yehia explored the realms of fuzzy ideals and fuzzy subalgebras in [33]. Subsequently, the realm of Lie algebras has been extended to include fuzzy frameworks. In the literature review, we find several recent contributions closely related to our research. For example, Yehia's work on fuzzy ideals and fuzzy subalgebras of Lie algebras [33], provides a foundation for understanding the concept of fuzzy sets and systems. El-Bably et al. [16]'s study on topological reduction for predicting lung cancer disease, has been influential in enhancing prediction models for diseases.

El-Bably and El-Atik [15]'s exploration of soft β-rough sets, has been immensely valuable in discerning COVID-19, a subject of profound significance in current times. Moreover, El-Gayar et al. [18]'s research on economic decision-making employing rough topological structures , has expanded our comprehension of economic applications. In addition, Yusoff et al. [34]'s study on the circular q-rung orthopair fuzzy set and its algebraic properties contributes to further advancements in this field.

In the realm of medical diagnosis, Abu-Gdairi and collaborators have delved into the topological visualization and graph analysis of rough sets concerning human heart data [1]. El-Bably et al. introduced novel topological approaches to generalized soft rough approximations in medical applications [14], and their exploration of medical diagnosis for Chikungunya disease using soft rough sets has notably contributed to the field [13]. Atef et al.'s investigation into covering soft rough sets and their topological properties has further expanded the scope of this domain [10]. Furthermore, recent years have seen extensive research driven by the potential applications of fuzzy sets across diverse domains. Jun et al. notably applied fuzzy soft set theory to BCK/BCI-algebras [24], while Muhiuddin et al. explored generalized ideals of BCK/BCI-Algebras based on fuzzy soft set theory [30], making significant strides in this area [31].

Moreover, Abu-Gdairi et al. [2] have explored the application of fuzzy point concepts to fuzzy topological spaces, opening up new possibilities for understanding topological structures. El-Bably and El-Sayed [17], have introduced innovative methods to generalize Pawlak approximations using simply open concepts, with specific applications in economics. The ongoing challenges posed by the COVID-19 pandemic have motivated research on topological models of rough sets and their application to decision-making, as demonstrated by El-Gayar and El-Atik [19]. Lu et al. [26] have introduced a novel type of generalized picture fuzzy soft set with practical applications in decision-making processes.

Furthermore, Ali et al. [6] have proposed a topological approach to generalized soft rough sets via near concepts, contributing to the advancement of this field. Additionally, Abd El-Monsef et al. [20] conducted a comparative analysis of three types of rough fuzzy sets based on two universal sets, providing valuable insights. These recent contributions have expanded the boundaries of fuzzy set theory and its applications, laying the groundwork for further exploration and research in this domain. On the other hand, the study on n-polar fuzzy Lie ideals within the context of Lie algebras, exploring their diverse properties related to nilpotency, is complemented by the introduction of n-polar fuzzy adjoint representations of Lie algebras and their correlation with nilpotent n-polar fuzzy Lie ideals. Additionally, recent contributions, such as the exploration of m-polar fuzzy ideals of BCK/BCI-algebras [5] have significantly widened the scope and addressed specific challenges within this field.

This paper organization is as follows: In Section 2, we offer an overview of pertinent studies that have influenced our research. In Section 3, we provide a comprehensive definition of the n-polar Z-hesitant anti-FSo set for subalgebras. Moving on to Section 4, we introduce the concept of n-polar Z-hesitant anti-fuzzy ideals specifically within the context of BCK/BCI-algebras. Moving on to Section 5, we investigate the notion of a closed n-polar Z-hesitant anti-FSo ideal. Section 6 is dedicated to discussing the concept of an n-polar Z-hesitant anti-FSo commutative ideal. Finally, the last section offers a conclusion and outlines our future research ideas.

2 Preliminaries

In this section, we revisit fundamental definitions that serve as the bedrock of our research.

Definition 2.1. *An algebra denoted as $(X; *, 0)$ of type $(2, 0)$ is defined as a BCK-algebra if it satisfies the following conditions:*

1. $(\forall i, j, k \in X)\Big(\big((i * j) * (i * k)\big) * (k * j) = 0.$
2. $(\forall i, j \in X)\Big(\big(i * (i * j)\big) * j = 0\Big).$
3. $(\forall i \in X)\big(i * i = 0\big).$
4. $(\forall i, j \in X)\big(i * j = 0, j * i = 0 \to i = j\big).$
5. $(\forall i \in X)\big(0 * i = 0\big).$

Then, X is referred to as a BCK-algebra. Any BCK-algebra X must adhere to the following axioms:

6. $(\forall i \in X)\big(i * 0 = i\big).$
7. $(\forall i, j, k \in X)\big(i \le j \to i * k \le j * k, k * j \le k * i\big).$
8. $(\forall i, j, k \in X)\big((i * j) * k = (i * k) * j\big).$
9. $(\forall i, j, k \in X)\Big(\big(i * k\big) * \big(j * k\big) \le i * j\Big)$ *where $i \le j$ if and only if $i * j = 0$.*

In addition, any BCI-algebra X must satisfy the following additional axiom:

1. $(\forall i, j, k \in X)\Big(0 * \Big(0 * \big((i * k) * (j * k)\big)\Big)\Big) = \big(0 * j\big) * \big(0 * i\big).$

Definition 2.2. [21] *A non-empty subset S of a BCK/BCI-algebra X is considered a subalgebra if, for all $i, j \in S$, $i * j \in S$.*

Definition 2.3. [21] *A non-empty subset I of a BCK/BCI-algebra X is called an ideal if it satisfies the following conditions:*

ID1: $0 \in I.$

ID2: $(\forall a, b \in X)(a * b \in I, a \in I \to b \in I).$

Definition 2.4. [22] *Consider X as a BCK/BCI-algebra. We can represent a hesitant FS on X as H, and denote it as follows: $H := \{(a, \mu_H(a)) | a \in X\}$, is termed a hesitant fuzzy subalgebra if it satisfies the following condition:*

$$(\forall a, b \in X)\Big(\mu_H(a * b) \supseteq \mu_H(a) \cap \mu_H(b)\Big). \tag{1}$$

Definition 2.5. [22] *Assuming X is a BCK/BCI-algebra, we can define a hesitant FS on X as H, represented as $H := (a, \mu_H(a)) | a \in X$. It qualifies as a hesitant fuzzy ideal if it meets the following requirement:*

$$(\forall a, b \in X)\Big(\mu_H(a * b) \cap \mu_H(b) \subseteq \mu_H(a) \subseteq \mu_H(0)\Big). \tag{2}$$

Definition 2.6. [23] *Given a subset λ of P, a hesitant FSo set denoted as (H, λ) over the set X is termed a hesitant FSo subalgebra with respect to $e \in \lambda$ if the hesitant FS $H_{[e]} := (a, \mu_{H_{[e]}}(a)) | a \in X$ establishes itself as a hesitant fuzzy subalgebra within the context of X.*

Definition 2.7. [23] *When considering a subset λ of P, a hesitant FSo set denoted as (H, λ) over the set X earns the title of a hesitant FSo ideal based on $e \in \lambda$ if the hesitant FS $H_{[e]} := (a, \mu_{H_{[e]}}(a)) | a \in X$ is recognized as a hesitant fuzzy subalgebra of X.*

Definition 2.8. [7] *Let X be a non-empty finite universe and Z be a non-empty set. A Z-hesitant FS H_Z is a set defined as:*

$$H_Z = \Big\{\Big((a,z), \mu_{H_Z}(a,z)\Big) \mid a \in X, z \in Z\Big\}, \tag{3}$$

where $\mu_{H_Z} : X \times Z \to [0,1]$.

Definition 2.9. *For a subset λ of P, let X be a non-empty finite universe and Z be a non-empty set. The Z-hesitant FSo set based on $e \in \lambda$ is defined as:*

$$H_{Z_{[e]}} = \Big\{\Big((a,z), \mu_{H_{Z_{[e]}}}(a,z)\Big) \mid a \in X, z \in Z\Big\}, \tag{4}$$

where $\mu_{H_{Z_{[e]}}} : X \times Z \to [0,1]$.

Definition 2.10. *An n-polar Z-hesitant FS on a non-empty set X is a mapping $H_Z : X \times Z \to [0,1]^n$. The membership value of every element $a \in X$ is denoted by:*

$$H_Z = \Big\{\Big((a_1,z), \mu_{H_Z}(a_1,z)\Big), \Big((a_2,z), \mu_{H_Z}(a_2,z)\Big), \ldots, \Big((a_n,z), \mu_{H_Z}(a_n,z)\Big)\Big\}, \tag{5}$$

which we can write as:

$$H^i = \Big\{\Big((a,z), \mu^i_{H_Z}(a,z)\Big) \mid a \in X, z \in Z\Big\}, \tag{6}$$

for all $i = 1,2,\ldots,n$, where $H^i : [0,1]^n \to [0,1]$ is the i-th projection.

3 n-polar Z-hesitant Anti-fuzzy Soft Subalgebra

In this section, we present a new concept of n-polar Z-hesitant anti-FSo set, its theorems, and some of its fundamental properties.

Definition 3.1. *Let X be a BCK/BCI-algebra. We define the n-polar Z-hesitant anti-FS H^i on X as follows:*

$$H^i = \Big\{\Big((a,z), \mu^i_{H_Z}(a,z)\Big) \mid a \in X, z \in Z\Big\}. \tag{23}$$

This collection is denoted as an n-polar Z-hesitant anti-fuzzy subalgebra when, for each $i = 1,2,\ldots,n$ it fulfills the following criterion:

$$\forall\, a,b \in X, z \in Z : \mu_{H^i}(a * b, z) \subseteq \mu_{H^i}(a,z) \cup \mu_{H^i}(b,z). \tag{24}$$

Definition 3.2. *Consider a set of elements P. For any subset λ of P, we define an n-polar Z-hesitant anti-FSo set (H^i, λ). This set is termed an n-polar Z-hesitant anti-FSo subalgebra based on an element $e \in \lambda$ if the n-polar Z-hesitant anti-FS H^i on X, given by:*

$$H^i = \Big\{\Big((a,z), \mu^i_{H_Z}(a,z)\Big) \mid a \in X, z \in Z\Big\}. \tag{25}$$

Is considered a Z-hesitant anti-FSo subalgebra within the scope of X for all $i = 1,2,\ldots,n$.

Example 3.1. *Let $X = \{\aleph_1, \aleph_2, \aleph_3\}$ be a BCK-algebra set, and consider the operation $*$ on X defined in the following Table 1:*

Table 1: Operation $*$ on set X.

$*$	$\aleph_1$	$\aleph_2$	$\aleph_3$
$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\aleph_2$	$\aleph_2$	$\aleph_1$	$\aleph_2$
$\aleph_3$	$\aleph_3$	$\aleph_3$	$\aleph_1$

Then, $(X, *, \aleph_1)$ is a BCK-algebra.

Consider the set $Z = \{\varphi\}$, and a element set $Z = \{e_1, e_2\}$, let $n = 2$, which is described in the following Table 2:

Table 2: 2-polar Z-hesitant anti-FSo subalgebra with respect to elements in X.

X	$(\aleph_1, \varphi)$	$(\aleph_2, \varphi)$	$(\aleph_3, \varphi)$
e_1	$(0.9, 0.9)(0.7)$	$(0.5, 0.3)(0.1, 0.3)$	$(0.3, 0.3)(0.5, 0.3)$
e_2	$(0.5, 0.6, 0.5)(0.9, 0.9)$	$(0.4)(0.1, 0.1, 0.1)$	$(0.2)(0.1, 0.3)$

The table presented earlier demonstrates that, with respect to the given elements, it constitutes a 2-polar Z-hesitant anti-FSo subalgebra within the context of X.

Proposition 3.1. *If (H^i, λ) is an n-polar Z-hesitant anti-FSo subalgebra over X, then for all $a \in X$ and $z \in Z$,*

$$\mu_{H^i_{[e]}}(a, z) \supseteq \mu_{H^i_{[e]}}(0, z). \tag{7}$$

In this context, e can be any element within the set λ, while i varies over the range from 1 to n.

Proof. For every a in X and e within λ,

$$\begin{aligned}\mu_{H^i_{[e]}}(0, z) &= \mu_{H^i_{[e]}}(a * a, z) \\ &\subseteq \mu_{H^i_[e]}(a, z) \cup \mu_{H^i_{[e]}}(a, z) \\ &= \mu_{H^i_{[e]}}(a, z).\end{aligned}$$

Here, with z belonging to set Z and i spanning from 1 to n. This concludes the proof. □

Proposition 3.2. *If every n-polar Z-hesitant anti-FSo subalgebra of X adheres to the following inequality:*

$$\forall a, b \in X, \forall z \in Z : \mu_{H^i[e]}(a * b, z) \subseteq \mu H^i_{[e]}(a, z), \tag{8}$$

for all $i = 1, 2, \ldots, n$, then it follows that:

$$\mu_{H^i[e]}(a, z) = \mu H^i_{[e]}(0, z). \tag{9}$$

Proof. By using $\forall a \in X : a * 0 = a$ from BCI/BCK-algebra definition. Then, let $a \in X$, $z \in Z$, and $e \in \lambda$, we have,

$$\begin{aligned}\mu_{H^i_{[e]}}(a, z) &= \mu_{H^i_{[e]}}(a * 0, z) \\ &\subseteq \mu_{H^i_{[e]}}(0, z).\end{aligned} \tag{10}$$

It follows from the previous proposition that,

$$\mu_{H^i_{[e]}}(a,z)=\mu_{H^i_{[e]}}(0,z). \tag{11}$$

□

4 n-polar Z-hesitant Anti-fuzzy Soft Ideal

Let's start by defining the concept of a Z-hesitant fuzzy ideal.

Definition 4.1. *Let,*

$$H^i=\Big\{(a,z),\mu_{H^i_Z}(a,z)\mid a\in X,z\in Z\Big\}, \tag{12}$$

be a Z-hesitant anti-FS in X. Then H^i is called an n-polar Z-hesitant anti-fuzzy ideal of X if it satisfies the following conditions:

1. $\mu_{H^i_Z}(0,z)\subseteq\mu_{H^i_Z}(a,z)$ *for all* $a\in X$, $z\in Z$.
2. $\mu_{H^i_Z}(a,z)\subseteq\mu_{H^i_Z}(a*b,z)\cup\mu_{H^i_Z}(b,z)$ *for all* $a,b\in X$, $z\in Z$, *and* $i=1,2,\ldots,m$.

Definition 4.2. *Consider (H^i,λ) as a hesitant anti-FSo set over X, with λ denoting a subset of the element set P. If $e\in\lambda$, we designate (H^i,λ) as an n-polar Z-hesitant anti-FSo ideal, based on the presence of e in λ, provided that the n-polar Z-hesitant anti-FS,*

$$H^i=\Big\{(a,z),\mu_{H^i_Z}(a,z)\mid a\in X,z\in Z\Big\}. \tag{13}$$

Is considered a hesitant fuzzy ideal within the scope of X for all $i=1,2,\ldots,n$.

Example 4.1. *Let $X=\{\aleph_1,\aleph_2\}$ be the BCK-algebra set. We consider the operation $*$ defined in the following Table 3:*

Table 3: Operation $*$ on set X.

$*$	$\aleph_1$	$\aleph_2$
$\aleph_1$	$\aleph_1$	$\aleph_1$
$\aleph_2$	$\aleph_2$	$\aleph_1$

Then $(X,*,\aleph_1)$ is a BCK-algebra.

Now, define the set $Z=\{\alpha,\beta\}$, and the element set is $Z=\{p_1,p_2,p_3\}$ with a 2-polar anti-FS on X, which is described in the following Table 4:

Table 4: Description of 2-polar Z-hesitant anti-FSo ideal.

	$(\aleph_1,\alpha)$	$(\aleph_1,\beta)$	$(\aleph_2,\alpha)$	$(\aleph_2,\beta)$
p_1	(0.9, 0.8) (0.8, 0.8)	(0.7, 0.5, 0.3) (0.5, 0.5)	(0.6, 0.9, 0.3)([0.1, 0.8)	(0.7, 0.4) (0.5, 0.3, 0.1)
p_2	(0.9) (0.7, 0.7)	(0.8, 0.8) (0.9, 0.6, 0.7)	(0.6, 0.3, 0.9) (0.6, 0.7)	(0.3, 0.8) (0.4, 0.9)
p_3	(0.8) (0.8, 0.8)	(0.7, 0.7) (0.8)	(0.5, 0.8) (0.1, 0.8)	(0.2, 0.7) (0.4, 0.8)

Thus, it's a 2-polar Z-hesitant anti-FSo ideal.

Theorem 4.1. *For any BCK-algebra X, every n-polar Z-hesitant anti-FSo ideal is a n-polar Z-hesitant anti-FSo subalgebra.*

Proof. Let $e \in \lambda$, $z \in Z$, and (H^i, λ) be a n-polar Z-hesitant anti-FSo ideal over X. Then,

$$\begin{aligned}\mu_{H^i_{[e]}}(a * b, z) &\subseteq \mu_{H^i_{[e]}}\big((a * b) * a, z\big) \cup \mu_{H^i_{[e]}}(a, z)\\ &= \mu_{H^i_{[e]}}\big((a * a^{-1}) * b, z\big) \cup \mu_{H^i_{[e]}}(a, z)\\ &= \mu_{H^i_{[e]}}(0 * b, z) \cup \mu_{H^i_{[e]}}(a, z)\\ &= \mu_{H^i_{[e]}}(0, z) \cup \mu_{H^i_{[e]}}(a, z)\\ &\subseteq \mu_{H^i_{[e]}}(b, z) \cup \mu_{H^i_{[e]}}(a, z).\end{aligned}$$

For every $a, b \in X$, $z \in Z$, and for each $i = 1, 2, \ldots, n$, it follows that (H^i, λ) constitutes an n-polar Z-hesitant anti-FSo subalgebra within the domain of X. This concludes the proof. □

Proposition 4.1. *All n-polar Z-hesitant anti-FSo ideals, denoted as (H^i, λ), within the context of X, adhere to the following:*

$$\forall e \in \lambda, \forall z \in Z, \forall a, b, c \in X : (a, b \leq c - \epsilon)\, \mu_{H^i_{[e]}}(a, z) \subseteq \mu_{H^i_{[e]}}(b, z) \cup \mu_{H^i_{[e]}}(c, z). \tag{14}$$

Proof. Let $e \in \lambda$, $z \in Z$, and $a, b, c \in X$ such that $a \cdot b \leq c$. Then, $(a * b) * c = 0$, and so:

$$\begin{aligned}\mu_{H^i_{[e]}}(a * b, z) &\subseteq \mu_{H^i_{[e]}}\big((a * b) * c, z\big) \cup \mu_{H^i_{[e]}}(c, z)\\ &= \mu_{H^i_{[e]}}(0, z) \cup \mu_{H^i_{[e]}}(c, z)\\ &= \mu_{H^i_{[e]}}(c, z),\end{aligned}$$

it follows that,

$$\mu_{H^i_{[e]}}(a, z) \subseteq \mu_{H^i_{[e]}}(a * b, z) \cup \mu_{H^i_{[e]}}(b, z) \subseteq \mu_{H^i_{[e]}}(b, z) \cup \mu_{H^i_{[e]}}(c, z).$$

This completes the proof. □

Proposition 4.2. *Every n-polar Z-hesitant anti-FSo ideal over BCI-algebra X satisfies the following:*

$$\forall e \in \lambda, \forall z \in Z, \forall a \in X : \mu_{H^i_{[e]}}\big(0 * (0 * a), z\big) \subseteq \mu_{H^i_{[e]}}(a, z). \textit{ For all } i = 1, 2, \ldots m. \tag{15}$$

Proof. Let (H^i, λ) be a n-polar Z-hesitant anti-FSo ideal. Then, for $e \in \lambda$, $z \in Z$, and $a \in X$, we have:

$$\begin{aligned}\mu_{H^i_{[e]}}(0 * (0 * a), z) &\subseteq \mu_{H^i_{[e]}}\Big(\big(0 * (0 * a)\big) * a, z\Big) \cup \mu_{H^i_{[e]}}(a, z)\\ &= \mu_{H^i_{[e]}}(0, z) \cup \mu_{H^i_{[e]}}(a, z)\\ &= \mu_{H^i_{[e]}}(a, z).\end{aligned}$$

With this, we conclude the proof. □

Proposition 4.3. *Each n-polar Z-hesitant anti-FSo ideal satisfies the following conditions:*
For all $a, b, c \in X$, for all $z \in Z$ and $e \in \lambda$:

1. *If* $a \leq b$, *then* $\mu_{H^i_{[e]}}(a, z) \subseteq \mu_{H^i_{[e]}}(b, z)$.
2. $\mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}(a * c, z) \cup \mu_{H^i_{[e]}}(c * b, z)$.
3. *If* $\mu_{H^i_{[e]}}(a * b, z) = \mu_{H^i_{[e]}}(0, z)$, *then* $\mu_{H^i_{[e]}}(a, z) \subseteq \mu_{H^i_{[e]}}(b, z)$.

Proof. Assume $e \in \lambda$, $z \in Z$, and $a, b, c \in X$:

1. We have $a \leq b$, then $a * a^{-1} = 0$ because (H^i, λ) is a n-polar Z-hesitant anti-FSo ideal of X. As a result:
$$\begin{aligned}\mu_{H^i_{[e]}}(a, z) &\subseteq \mu_{H^i_{[e]}}(a * b, z) \cup \mu_{H^i_{[e]}}(b, z)\\ &\subseteq \mu_{H^i_{[e]}}(b, z).\end{aligned}$$
2. From $(a * b) * (a * c) \not\leq c * b$, we get:
$$\mu_{H^i_{[e]}}\big((a * b) * (a * c), z\big) \subseteq \mu_{H^i_{[e]}}(c * b, z). \text{ Therefore,}$$
$$\begin{aligned}\mu_{H^i_{[e]}}(a * b, z) &\subseteq \mu_{H^i_{[e]}}\big((a * b) * (a * c), z\big) \cup \mu_{H^i_{[e]}}(a * c, z)\\ &\subseteq \mu_{H^i_{[e]}}(a * c, z) \cup \mu_{H^i_{[e]}}(c * b, z).\end{aligned}$$
$$\begin{aligned}&\mu_{H^i_{[e]}}\big((a * b) * (a * c), z\big) \subseteq \mu_{H^i_{[e]}}(c * b, z)\\ &\quad\Rightarrow \mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}(a * c, z) \cup \mu_{H^i_{[e]}}(c * b, z).\end{aligned}$$
3. If $\mu_{H^i_{[e]}}(a * b, z) = \mu_{H^i_{[e]}}(0, z)$, then:
$$\begin{aligned}\mu_{H^i_{[e]}}(a, z) &\subseteq \mu_{H^i_{[e]}}(a * b, z) \cup \mu_{H^i_{[e]}}(b, z)\\ &= \mu_{H^i_{[e]}}(0, z) \cup \mu_{H^i_{[e]}}(b, z)\\ &= \mu_{H^i_{[e]}}(b, z).\end{aligned}$$

With this, we conclude the proof. □

5 Closed n-polar Z-hesitant Anti-FSo Ideal

In this section we explore the idea of Closed n-polar Z-hesitant anti-FSo ideal.

Definition 5.1. *A n-polar Z-hesitant anti-fuzzy ideal,*
$$H^i = \Big\{(a, z), \mu^i_H(a, z) \mid a \in X, z \in Z\Big\}. \tag{16}$$
In the context of a BCI-algebra, we consider it closed when
$$\mu^i_H(a, z) \supseteq \mu^i_H(0 * a, z), \tag{17}$$
for all $a \in X$, $z \in Z$, and $i = 1, 2, \ldots, n$.

Definition 5.2. *An n-polar Z-hesitant anti-FSo ideal, denoted as (H^i, λ), established over a BCI-algebra X and centered around a element $e \in \lambda$, is deemed closed if the hesitant anti-FS,*

$$H^i_{[e]} = \Big\{(a,z), \mu^i_{H_{[e]}}(a,z) \mid a \in X, z \in Z\Big\}. \tag{18}$$

Is regarded as a closed hesitant anti-fuzzy ideal within the scope of X for all $i = 1, 2, \ldots, n$.

Example 5.1. *Let $X = \{\aleph_1, \aleph_2\}$ be the BCK-algebra set with a binary operation "$*$", which is given in the following Table 5:*

Table 5: Operation $*$ on set X.

$*$	$\aleph_1$	$\aleph_2$
$\aleph_1$	$\aleph_1$	$\aleph_1$
$\aleph_2$	$\aleph_2$	$\aleph_1$

Now, let $Z = \{A\}$ with a 2-polar anti-FS in X. Then the elements set is defined as $Z = \{e_1, e_2\}$ in Table 6.

Table 6: Description of closed n-polar Z-hesitant anti-FSo ideal.

	$(\aleph_1, A)$	$(\aleph_2, A)$
e_1	$[0.9]$ $(0.8, 0.7)$ $(0.7, 0.7)$	$(0.4, 0.9)$ $(0.3, 0.8, 0.4)$ $(0.6, 0.7, 0.2)$
e_2	(0.5) $(0.6, 0.7, 0.9)$ $(0.8, 0.8)$	$(0.5, 0.1)$ $(0.3, 0.9, 0.5)$ $(0.8, 0.2)$

Then it's a closed n-polar Z-hesitant anti-FSo ideal.

Theorem 5.1. *Consider (H^i, λ) as an n-polar Z-hesitant anti-FSo set. Then, a closed n-polar Z-hesitant anti-FSo ideal within the BCI-algebra X, which is centered around a specific element, is essentially an n-polar Z-hesitant anti-FSo subalgebra over X that shares the same element.*

Proof. Consider (H^i, λ) be a closed n-polar Z-hesitant anti-FSo ideal over X based on $e \in \lambda$. Then,

$$\mu_{H^i_{[e]}}(a,z) \supseteq \mu_{H^i_{[e]}}(0 * a, z).$$

When considering all $a, b \in X$, $z \in Z$, and $i = 1, 2, \ldots, m$, it can be deduced that,

$$\mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}\big((a * b) * a, z\big) \cup \mu_{H^i_{[e]}}(a,z) \subseteq \mu_{H^i_{[e]}}(a,z) \cup \mu_{H^i_{[e]}}(b,z),$$

for all $a, b \in X$, $z \in Z$. Hence, (H^i, λ) qualifies as an n-polar Z-hesitant anti-FSo subalgebra within the context of X, based on the presence of the element $e \in \lambda$ for all $i = 1, 2, \ldots, n$. □

Theorem 5.2. *Let (H^i, λ) represent an n-polar Z-hesitant anti-FSo ideal within a BCI-algebra X, based on a element $e \in \lambda$. We designate it as 'closed' if and only if it satisfies the following condition:*

$$\forall a, b \in X, z \in Z : \mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}(a,z) \cup \mu_{H^i_{[e]}}(b,z). \tag{19}$$

Proof. $(\Rightarrow)$ We start by assuming that (H^i, λ) is a closed n-polar Z-hesitant anti-FSo ideal in a BCI-algebra X based on a element $e \in \lambda$. Consider the fact that for all $a, b \in X$, we have $a * b \leq 0 * b$. Therefore, we can establish the following relationship:

$$\mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}(a,z) \cup \mu_{H^i_{[e]}}(0 * b, z) \subseteq \mu_{H^i_{[e]}}(a,z) \cup \mu_{H^i_{[e]}}(b,z).$$

This holds true for all $a, b \in X$, $z \in Z$, and $i = 1, 2, \ldots, n$.

($\Leftarrow$) On the contrary, (H^i, λ) be an n-polar Z-hesitant anti-FSo ideal over a BCI-algebra X based on a element $e \in \lambda$. Given that,

$$\mu_{H^i_{[e]}}(a, z) \supseteq \mu_{H^i_{[e]}}(0, z),$$

for all $a \in X$, $z \in Z$, we can deduce that,

$$\mu_{H^i_{[e]}}(0 * a, z) \subseteq \mu_{H^i_{[e]}}(0, z) \cup \mu_{H^i_{[e]}}(a, z) = \mu_{H^i_{[e]}}(a, z).$$

This relationship holds for all $a \in X$, $z \in Z$. As a result, (H^i, λ) represents a closed n-polar Z-hesitant anti-FSo ideal within a BCI-algebra X, contingent on the presence of the element $e \in \lambda$, for all $i = 1, 2, \ldots, n$. □

6 n-polar Z-hesitant Anti-FSo Commutative Ideal

In a previous study conducted by Jun and Meng [25], the concept of commutative ideals was introduced. In this section, we will extend this concept by introducing the notion of an n-polar Z-hesitant FSo commutative ideal. We will proceed to establish and confirm various properties and theorems associated with this concept. To begin, let's provide a formal definition of the n-polar Z-hesitant anti-fuzzy commutative ideal.

Definition 6.1. *In the context of a BCK-algebra, we define a n-polar Z-hesitant FS H^j as follows:*

$$H^j = \Big\{(a, z) \mid \mu^j_H(a, z) \text{ for } a \in X, z \in Z\Big\}. \tag{20}$$

H^j is termed an n-polar Z-hesitant anti-fuzzy commutative ideal of X if it adheres to the following conditions:

1. *$\mu^j_H(0, z) \subseteq \mu_j H_b(a, z)$ for all $a \in X$.*
2. *$\mu^j_H(a * (\gamma * (\gamma * a)), z) \subseteq \mu^j_H((a * \gamma) * c, z) \cup \mu^j_H(c, z)$ for all $a, \gamma, c \in X$ and $j = 1, 2, \ldots, n$.*

Definition 6.2. *Consider a n-polar Z-hesitant anti-FSo set over X, denoted as (H^i, λ), where λ forms a subset of the elements set P. If $e \in \lambda$, then we define (H^i, λ) as a n-polar Z-hesitant anti-FSo commutative ideal. This definition hinges on the n-polar Z-hesitant anti-FS $H^i_{[e]}$, which is expressed as:*

$$H^i_{[e]} := \Big\{(a, z) \mid \mu^i_{H_{[e]}}(a, z) \text{ for } a \in X, z \in Z\Big\}. \tag{21}$$

Moreover, it must meet the conditions of being a hesitant anti-fuzzy commutative ideal within the domain of X for all $i = 1, 2, \ldots, n$.

Example 6.1. *Consider the set $X = \{\aleph_1, \aleph_2, \aleph_3\}$ representing a BCK-algebra set. Let's define the set $Z = \{\xi\}$ and introduce a set of elements denoted as $E = \{e_1, e_2, e_3\}$. We also have a 2-polar anti-FS in X. Initially, we define the binary operation $*$ on X as follows using the Cayley Table 7:*

Table 7: Operation $*$ on set X.

$*$	$\aleph_1$	$\aleph_2$	$\aleph_3$
$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\aleph_2$	$\aleph_2$	$\aleph_1$	$\aleph_2$
$\aleph_3$	$\aleph_3$	$\aleph_3$	$\aleph_1$

It shows that $(X, *, \aleph_1)$ is a BCK-algebra. Let:

Table 8: Description of 2-polar Z-hesitant anti-FSo commutative ideal.

	$(\aleph_1, \xi)$	$(\aleph_2, \xi)$	$(\aleph_3, \xi)$
e_1	(0.9) (0.8 0.9)	(0.4 0.3 0.9) (0.6 0.9)	(0.3 0.9) (0.1 0.9 0.4)
e_2	(0.8) (0.9)	(0.8 0.1)(0.2 0.9)	(0.6 0.8) (0.5 0.9 0.3)
e_3	(0.8) (0.7)	(0.3 0.8) (0.1 0.7 0.1)	(0.3 0.8) (0.4 0.7 0.3 0.1)

The previous Table 8 shows that it's a 2-polar Z-hesitant anti-FSo commutative ideal.

Theorem 6.1. *Every n-polar Z-hesitant anti-FSo commutative ideal also qualifies as an n-polar Z-hesitant anti-FSo ideal of X.*

Proof. For any a, b, c in X, q in Z, and e in λ, let's consider (H^i, λ) as a n-polar Z-hesitant anti-FSo commutative ideal of X.

$$\begin{aligned}\mu_{H^i_{[e]}}(a, z) &= \mu_{H^i_{[e]}}\big(a * (0 * (0 * a)), z\big)\\ &\subseteq \mu_{H^i_{[e]}}\big((a * 0), z\big) \cup \mu_{H^i_{[e]}}(c, z) \cup \mu_{H^i_{[e]}}(c * c, z) \cup \mu_{H^i_{[e]}}(c, z).\end{aligned}$$

This holds for all a, c in X, q in Z, and e in λ. Consequently, (H^i, λ) qualifies as an n-polar Z-hesitant anti-FSo ideal. □

Theorem 6.2. *Suppose (H^i, λ) represents a n-polar Z-hesitant anti-FSo ideal. Then, it is identified as a n-polar Z-hesitant anti-FSo commutative ideal if and only if it adheres to the condition:*

$$\mu_{H^i_{[e]}}\big(a * (b * (a * b)), z\big) \subseteq \mu_{H^i_{[e]}}(a * b, z), \tag{22}$$

for all a, b in X, q in Z, and e in λ.

Proof. Consider $a, b \in X$, $z \in Z$, and $e \in \lambda$,

$(\Rightarrow)$ (H^i, λ) being a n-polar Z-hesitant anti-FSo commutative ideal of X implies that when $c = 0$, we have

$$\mu_{H^i_{[e]}}\big(a * (b * (a * b)), z\big) \subseteq \mu_{H^i_{[e]}}(a * b * 0, z) \cup \mu_{H^i_{[e]}}(0, z) = \mu_{H^i_{[e]}}(a * b, z).$$

$(\Leftarrow)$ Conversely, if (H^i, λ) satisfies,

$$\mu_{H^i_{[e]}}\big(a * (0 * (0 * a)), z\big) \subseteq \mu_{H^i_{[e]}}(a * b, z),$$

then,

$$\mu_{H^i_{[e]}}(a * b, z) \subseteq \mu_{H^i_{[e]}}\big((a * b) * c, z\big) \cup \mu_{H^i_{[e]}}(c, z).$$

By combining these equations, we deduce that,

$$\mu_{H^i_{[e]}}\big(a * (b * (b * a)), z\big) \subseteq \mu_{H^i_{[e]}}\big((a * b) * c, z\big) \cup \mu_{H^i_{[e]}}(c, z),$$

valid for all $a, b, c \in X$, $z \in Z$, and $e \in \lambda$. Consequently, (H^i, λ) is a n-polar Z-hesitant anti-FSo commutative ideal. □

7 Conclusion

The concept of n-polar Z-hesitant anti-FSo sets in BCK-algebras plays a pivotal role in bridging the gap between algebraic structures and anti-FSo set theory, making a significant contribution to the field of abstract algebra.

While our research has shed light on the potential applications and extensions of n-polar Z-hesitant anti-FSo sets, it's important to acknowledge the limitations of our study. One limitation is the scope of our research sample, which may not encompass the full breadth of scenarios in which this concept could be applied. Future studies with larger and more diverse datasets could provide deeper insights.

Moreover, there is room for continuous exploration and advancement in this area. Several intriguing questions and avenues for further research have emerged from our study, including:

- The theory of FSo sets holds a significant place in the domain of fuzzy topological spaces. Can we extend this theory to introduce the notion of fuzzy BCK-algebra topological spaces, subsequently defining n-polar Z-hesitant anti-FSs in such spaces?
- Is it possible to establish the concept of fuzzy BCK-algebra ideal spaces as a precursor, paving the way for the definition of n-polar Z-hesitant anti-FSs in these ideal spaces?
- Exploring real-world applications of n-polar Z-hesitant anti-FSo sets in various domains, such as decision-making, data analysis, and beyond.

These questions not only underscore the current significance of n-polar Z-hesitant anti-FSo sets but also provide intriguing directions for future research and exploration within this field.

In summary, while our research has expanded our understanding of n-polar Z-hesitant anti-FSo sets, the journey is far from over. It is through the continuous exploration of these concepts and their practical applications that we can unlock new possibilities and advance the field further.

List of Abbreviations

MZHAFSs	:	n-polar Z-hesitant Anti-Fuzzy Soft Sets
FS	:	Fuzzy Set
FSs	:	Fuzzy Sets
FSo	:	Fuzzy Soft

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